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ON CERTAIN PRODUCT OF BANACH MODULES

MOHAMMAD RAMEZANPOUR^{1*}

¹ *School of Mathematics and Computer Science, Damghan University, P. O. Box 36716, Damghan 41167, Iran.
 ramezanpour@du.ac.ir*

ABSTRACT. Let A and B be Banach algebras and let B be an algebraic Banach A -bimodule. Then the ℓ^1 -direct sum $A \times B$ equipped with the multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1 \cdot b_2 + b_1 \cdot a_2 + b_1b_2), \quad (a_1, a_2 \in A, b_1, b_2 \in B)$$

is a Banach algebra denoted by $A \bowtie B$. Module extension algebras, Lau product and also the direct sum of Banach algebras are the main examples satisfying this framework. We obtain characterizations of bounded approximate identities, spectrum, and topological center of this product. This provides a unified approach for obtaining some known results of both module extensions and Lau product of Banach algebras.

1. INTRODUCTION

Let A and B be Banach algebra and B is a Banach A -bimodule, we say that B is an algebraic Banach A -bimodule if

$$a \cdot (b_1b_2) = (a \cdot b_1)b_2, \quad (b_1b_2) \cdot a = b_1(b_2 \cdot a), \quad (b_1 \cdot a)b_2 = b_1(a \cdot b_2),$$

for each $b_1, b_2 \in B$ and $a \in A$. Then the Cartesian product $A \times B$ with the algebra multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1 \cdot b_2 + b_1 \cdot a_2 + b_1b_2),$$

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* Speaker.

and with the norm $\|(a, b)\| = \|a\| + \|b\|$ becomes a Banach algebra provided $\|a \cdot b\| \leq \|a\|\|b\|$, which we denote it by $A \bowtie B$. We note that if we identify $A \times \{0\}$ with A , and $\{0\} \times B$ with B , in $A \bowtie B$, then B is a closed ideal while A is a closed subalgebra of $A \bowtie B$, and $(A \bowtie B)/B$ is isometrically isomorphic to A . In other words, $A \bowtie B$ is a strongly splitting Banach algebra extension of A by B .

Besides giving a new method of constructing Banach algebras, the product \bowtie has relevance with the following known products.

- (a) (*Direct product of two Banach algebras*) Let A and B be Banach algebra. If we define $a \cdot b = b \cdot a = 0$, then B is an algebraic Banach A -bimodule and

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2).$$

Therefore, $A \bowtie B$ is the direct product $A \times_1 B$.

- (b) (*The module extensions*) Let X be a Banach A -bimodule. Define $x_1x_2 = 0$, then X is an algebraic Banach A -bimodule, and

$$(a_1, x_1)(a_2, x_2) = (a_1a_2, a_1 \cdot x_2 + x_1 \cdot a_2).$$

Therefore, $A \bowtie X$ is the module extension $A \oplus_1 X$. Module extensions are known as a rich source of (counter-)examples in various situations in abstract harmonic analysis and functional analysis, [5].

- (c) (*θ -Lau product of Banach algebras*) Let A and B be Banach algebra and $\theta \in \Delta(A)$, the set of all non zero multiplicative linear functional on A . Then B with a module actions given by $a \cdot b = b \cdot a = \theta(a)b$ is an algebraic Banach A -bimodule and

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, \theta(a_1)b_2 + \theta(a_2)b_1 + b_1b_2).$$

Thus $A \bowtie B$ is the θ -Lau product $A \theta \times B$. This product was introduced by Lau [2] for certain class of Banach algebras and followed by Sangani Monfared [4] for the general case. An elementary very familiar example is the case that $A = \mathbb{C}$ with θ as the identity character i that we get the unitization $B^\# = \mathbb{C} \theta \times B$ of B .

- (d) (*T -Lau product of Banach algebras*) Let A and B be Banach algebra and $T : A \rightarrow B$ be an algebra homomorphism with $\|T\| \leq 1$. Define $a \cdot b = T(a)b = b \cdot a$. Then B is an algebraic Banach A -bimodule and

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, T(a_1)b_2 + b_1T(a_2) + b_1b_2).$$

Thus $A \bowtie B$ is the T -Lau product $A T \times B$. This type of product was first introduced by Bhatt and Dabhi for the case where B is commutative and was extended by Javanshiri and Nemati for the general case [1].

The purpose of the present note is to determine the Gelfand space of $A \bowtie B$ which turns out to be non trivial even though $A \bowtie B$ need not be commutative and to discuss the topological center of $A \bowtie B$. These topics are central to the general theory of Banach algebras.

2. MAIN RESULTS

Let B be a Banach A -bimodule, we recall that B is called symmetric if $a \cdot b = b \cdot a$ for all $a \in A$ and $b \in B$.

We start with the following propositions which characterize the basic properties of $A \bowtie B$ in terms of A and B . These results extend related results in [1, 4].

Proposition 2.1. [3] *Let B be an algebraic Banach A -bimodule. Then $A \bowtie B$ is commutative if and only if B is a symmetric Banach A -bimodule and both A and B are commutative.*

Proposition 2.2. [3] *Let B be an algebraic Banach A -bimodule. Then (a_0, b_0) is an identity for $A \bowtie B$ if and only if a_0 is an identity for A , $b_0 \cdot a = a \cdot b_0 = 0$ for all $a \in A$ and $a_0 \cdot b + b_0 b = b \cdot a_0 + b b_0 = b$ for all $b \in B$.*

Proposition 2.3. [3] *Let B be an algebraic Banach A -bimodule. Then $\{(a_\alpha, b_\alpha)\}$ is a bounded left approximate identity for $A \bowtie B$ if and only if $\{a_\alpha\}$ is a bounded left approximate identity for A , $\|b_\alpha \cdot a\| \rightarrow 0$ for all $a \in A$ and $a_\alpha \cdot b + b_\alpha b \rightarrow b$ for all $b \in B$.*

The dual of the space $A \bowtie B$ can be identified with $A^* \times B^*$ in the natural way $(\varphi, \psi)(a, b) = \varphi(a) + \psi(b)$. The dual norm on $A^* \times B^*$ is of course the maximum norm $\|(\varphi, \psi)\| = \max\{\|\varphi\|, \|\psi\|\}$. The following result identifies the Gelfand space of $A \bowtie B$. This is a generalization of [1, Theorem 2.2] and [4, Proposition 2.4].

Proposition 2.4. [3] *Let B be an algebraic Banach A -bimodule. If $E := \{(\varphi, 0) : \varphi \in \Delta(A)\}$ and $F := \{(\varphi, \psi) : \varphi \in \Delta(A) \cup \{0\}, \psi \in \Delta(B), \text{ and } a \cdot \psi = \psi \cdot a = \varphi(a)\psi \forall a \in A\}$, then $\Delta(A \bowtie B) = E \cup F$.*

Corollary 2.5. *Let A and B be commutative Banach algebras and B is an algebraic Banach A -bimodule which is also symmetric. Then $A \bowtie B$ is semisimple if and only if both A and B are semisimple.*

Let X be a Banach A -bimodule, for $a \in A, x \in X, x^* \in X^*, a^{**} \in A^{**}$ and $x^{**} \in X^{**}$ we define

$$\begin{aligned} (x^{**} \circ a^{**})(x^*) &= x^{**}(a^{**} \circ x^*), & (a^{**} \circ x^{**})(x^*) &= a^{**}(x^{**} \circ x^*) \\ (a^{**} \circ x^*)(x) &= a^{**}(x^* \circ x), & (x^{**} \circ x^*)(a) &= x^{**}(x^* \circ a) \\ (x^* \circ x)(a) &= x^*(x \cdot a), & (x^* \circ a)(x) &= x^*(a \cdot x). \end{aligned}$$

Clearly, for each $a^{**} \in A^{**}$ and $x^{**} \in X^{**}$ the mappings $b^{**} \rightarrow b^{**} \circ x^{**} : A^{**} \rightarrow X^{**}$ and $y^{**} \rightarrow y^{**} \circ a^{**} : X^{**} \rightarrow X^{**}$ are w^* - w^* -continuous. The first topological centres of module actions of A on X may therefore be defined as

$$\begin{aligned} Z_A^1(X^{**}) &= \{x^{**} \in X^{**} : a^{**} \rightarrow x^{**} \circ a^{**} \text{ is } w^*\text{-}w^*\text{-continuous}\}, \\ Z_X^1(A^{**}) &= \{a^{**} \in A^{**} : x^{**} \rightarrow a^{**} \circ x^{**} \text{ is } w^*\text{-}w^*\text{-continuous}\}. \end{aligned}$$

If we consider $X = A$ with the natural A -bimodule structure then we obtain the first Arens product on A^{**} . In this case we write $Z^1(A^{**})$ instead $Z_A^1(A^{**})$. The Banach algebra A is called Arens regular if $Z^1(A^{**}) = A^{**}$.

To state our next result we note that if B is an algebraic Banach A -bimodule then B^{**} is an algebraic Banach A^{**} -bimodule, when A^{**} and B^{**} are equipped with their first Arens products.

Theorem 2.6. [3] *Let B be an algebraic Banach A -bimodule.*

(1) *Suppose that A^{**} , B^{**} , and $(A \bowtie B)^{**}$ are equipped with their first Arens products. Then*

$$(A \bowtie B)^{**} \cong A^{**} \bowtie B^{**} \quad (\text{isometric isomorphism}).$$

(2) *If B is Arens regular then*

$$Z^1((A \bowtie B)^{**}) = (Z^1(A^{**}) \cap Z_B^1(A^{**})) \times Z_A^1(B^{**})$$

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